

Parametrization of Quintessence and Its Potential

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Abstract

We develop a theoretical method of constructing the quintessence potential directly from the effective equation of state function $w(z)$, which describes the properties of the dark energy. We apply our method to four parametrizations of equation of state parameter and discuss the general features of the resulting potentials. In particular, it is shown that the constructed quintessence potentials are all in the form of a runaway type.

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Recent observations of type Ia supernovae suggest that the expansion of the universe is accelerating and that two-thirds of the total energy density exists in a dark energy component with negative pressure [1, 2]. In addition, measurements of the cosmic microwave background [3] and the galaxy power spectrum [4] also indicate the existence of the dark energy. The simplest candidate for the dark energy is a cosmological constant Λ , which has pressure $P_\Lambda = -\rho_\Lambda$. Specifically, a reliable model should explain why the present amount of the dark energy is so small compared with the fundamental scale (fine-tuning problem) and why it is comparable with the critical density today (coincidence problem). The cosmological constant suffers from both these problems. One possible approach to constructing a viable model for dark energy is to associate it with a slowly evolving and spatially homogeneous scalar field ϕ , called “quintessence” [5]-[9]. Such a model for a broad class of potentials can give the energy density converging to its present value for a wide set of initial conditions in the past and possess tracker behavior.

The dark energy is characterized by its equation of state parameter w , which is in general a function of redshift z in quintessence models. The quintessence potential $V(\phi)$ and the equation of state $w_\phi(z)$ may be reconstructed from supernova observations [10, 11, 12]. In this letter we develop a theoretical method of constructing the quintessence potential $V(\phi)$ directly from the dark energy equation of state function $w_\phi(z)$. We apply this method to four typical parametrizations which fit the data well [13]-[18] and discuss the general features of the resulting potentials. The typical behavior of the constructed potentials is found to be a runaway type.

We consider a spatially flat FRW universe which is dominated by the non-relativistic matter and a spatially homogeneous scalar field ϕ . The Friedmann equation can be written as

$$H^2 = \frac{1}{3M_{pl}^2}(\rho_m + \rho_\phi), \quad (1)$$

where $M_{pl} \equiv (8\pi G)^{-1/2}$ is the reduced Planck mass and ρ_m is the matter density. The energy density ρ_ϕ and pressure P_ϕ of the evolving scalar field ϕ are given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad (2)$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi), \quad (3)$$

respectively, where $V(\phi)$ is the scalar field potential. The corresponding equation of state parameter is now given by

$$w_\phi \equiv \frac{P_\phi}{\rho_\phi}. \quad (4)$$

Using Eqs. (2) and (3), we have

$$\frac{1}{2}\dot{\phi}^2 = \frac{1}{2}(1 + w_\phi)\rho_\phi, \quad (5)$$

$$V(\phi) = \frac{1}{2}(1 - w_\phi)\rho_\phi. \quad (6)$$

The evolution of quintessence field is governed by the equation of motion

$$\dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = 0, \quad (7)$$

which yields

$$\begin{aligned} \rho_\phi(z) &= \rho_{\phi 0} \exp \left[3 \int_0^z (1 + w_\phi) d \ln(1 + z) \right] \\ &\equiv \rho_{\phi 0} E(z), \end{aligned} \quad (8)$$

where z is the redshift which is given by $1 + z = a_0/a$ and subscript 0 denotes the value of a quantity at the redshift $z = 0$ (present). In terms of $w_\phi(z)$, the scalar field potential (6) can be written as a function of the redshift z :

$$V[\phi(z)] = \frac{1}{2}(1 - w_\phi)\rho_{\phi 0}E(z). \quad (9)$$

With the help of $\rho_m = \rho_{m0}(1 + z)^3$ and Eq. (8), the Friedmann equation (1) becomes

$$H(z) = H_0 \left[\Omega_{m0}(1 + z)^3 + \Omega_{\phi 0}E(z) \right]^{1/2}, \quad (10)$$

where $\Omega_{m0} \equiv \rho_{m0}/(3M_{pl}^2 H_0^2)$ and $\Omega_{\phi 0} \equiv \rho_{\phi 0}/(3M_{pl}^2 H_0^2)$. Using Eq. (5), we have

$$\frac{d\phi}{dz} = \mp \frac{(1 + w_\phi)^{1/2}}{(1 + z)H(z)} [\rho_\phi(z)]^{1/2}, \quad (11)$$

where the upper (lower) sign applies if $\dot{\phi} > 0$ ($\dot{\phi} < 0$). The sign in fact is arbitrary, as it can be changed by the field redefinition, $\phi \rightarrow -\phi$. So we choose the upper sign in the following discussions. Substituting Eqs. (8) and (10) into Eq. (11) gives

$$\frac{d\phi}{dz} = -\sqrt{3}M_{pl} \frac{(1 + w_\phi)^{1/2}}{(1 + z)} \left[1 + r_0(1 + z)^3 E^{-1}(z) \right]^{-1/2}, \quad (12)$$

where $r_0 \equiv \Omega_{m0}/\Omega_{\phi 0}$ is the energy density ratio of matter to quintessence at present time.

We define dimensionless quantities

$$\tilde{V} \equiv V/\rho_{\phi 0}, \quad \tilde{\phi} \equiv \phi/M_{pl}. \quad (13)$$

The construction equations (9) and (12) can then be written as

$$\tilde{V}[\phi(z)] = \frac{1}{2}(1 - w_\phi)E(z), \quad (14)$$

$$\frac{d\tilde{\phi}}{dz} = -\sqrt{3} \frac{(1 + w_\phi)^{1/2}}{(1 + z)} \left[1 + r_0(1 + z)^3 E^{-1}(z) \right]^{-1/2}, \quad (15)$$

which relate the quintessence potential $V(\phi)$ to the equation of state function $w_\phi(z)$. Given an effective equation of state function $w_\phi(z)$, the construction equations (14) and (15) will allow us to construct the quintessence potential $V(\phi)$.

Our method is new in that it relates directly the quintessence potential to the equation of state function, and so enables us to construct easily the potential without assuming its form. For instance, in the reconstruction method discussed in Ref. [10], the reconstruction equations relate the potential and the equation of state to measurements of the luminosity distance. The potential may thus be reconstructed by way of the luminosity distance from supernova data. Usually this can be done by assuming the form of a potential $V(\phi)$. The dark energy properties are well described by the effective equation of state parameter $w_\phi(z)$ which in general depends on the redshift z .

Let us now consider the following four cases [13]-[18]: a constant equation of state parameter and three two-parameter parametrizations.

Case I: $w_\phi = w_0$ (Ref. [13])

$$\tilde{V}(z) = \frac{1}{2}(1 - w_0)(1 + z)^{3(1+w_0)}, \quad (16)$$

$$\frac{d\tilde{\phi}}{dz} = -\sqrt{3} \frac{(1 + w_0)^{1/2}}{(1 + z)} \left[1 + r_0(1 + z)^{-3w_0} \right]^{-1/2}. \quad (17)$$

Case II: $w_\phi = w_0 + w_1 z$ (Ref. [14])

$$\tilde{V}(z) = \frac{1}{2}(1 - w_0 - w_1 z)(1 + z)^{3(1+w_0-w_1)} e^{3w_1 z}, \quad (18)$$

$$\frac{d\tilde{\phi}}{dz} = -\sqrt{3} \frac{(1 + w_0 + w_1 z)^{1/2}}{(1 + z)} \left[1 + r_0(1 + z)^{-3(w_0-w_1)} e^{-3w_1 z} \right]^{-1/2}. \quad (19)$$

Case III: $w_\phi = w_0 + w_1 \frac{z}{1+z}$ (Ref. [15, 16, 17])

$$\tilde{V}(z) = \frac{1}{2} \left(1 - w_0 - w_1 \frac{z}{1+z} \right) (1 + z)^{3(1+w_0+w_1)} e^{-3w_1 \frac{z}{1+z}}, \quad (20)$$

$$\frac{d\tilde{\phi}}{dz} = -\sqrt{3} \frac{(1 + w_0 + w_1 \frac{z}{1+z})^{1/2}}{(1 + z)} \left[1 + r_0(1 + z)^{-3(w_0+w_1)} e^{3w_1 \frac{z}{1+z}} \right]^{-1/2}. \quad (21)$$

Case IV: $w_\phi = w_0 + w_1 \ln(1 + z)$ (Ref. [18])

$$\tilde{V}(z) = \frac{1}{2} [1 - w_0 - w_1 \ln(1 + z)] (1 + z)^{3(1+w_0) + \frac{3}{2} w_1 \ln(1+z)}, \quad (22)$$

$$\frac{d\tilde{\phi}}{dz} = -\sqrt{3} \frac{[1 + w_0 + w_1 \ln(1 + z)]^{1/2}}{(1 + z)} \left[1 + r_0(1 + z)^{-3w_0 - \frac{3}{2} w_1 \ln(1+z)} \right]^{-1/2}. \quad (23)$$

We have numerically evaluated these equations. Fig. 1 shows the evolution of the energy density of the quintessence $\rho_\phi(z)$, where we choose $w_0 = -0.8$, $w_1 = 0.1$ and $r_0 = 3/7$. At low redshift, all models obey the same evolution law, but deviation from this is clearly visible at redshift $z > 1$. Fig. 2 shows the constructed quintessence potential $V(\phi)$, which is in the form of a runaway potential.

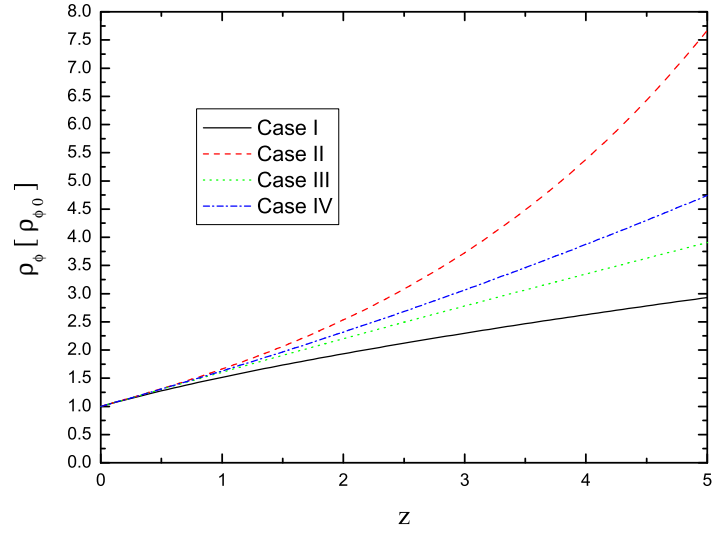


Figure 1: Evolution of the energy density of the quintessence $\rho_\phi(z)$.

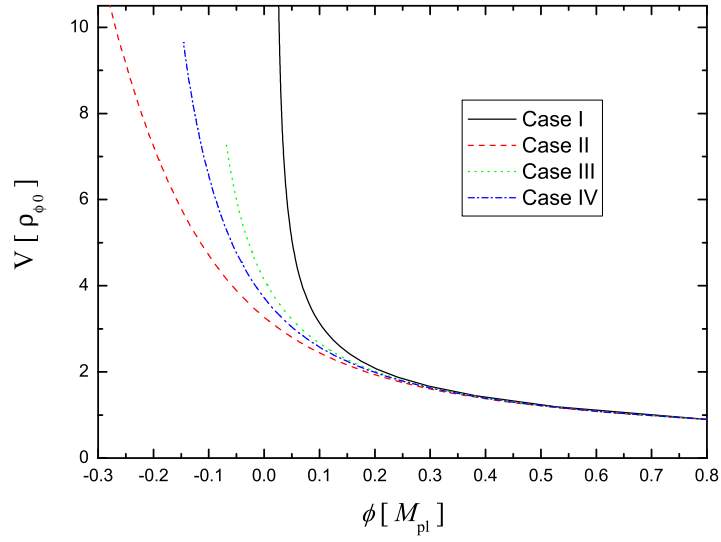


Figure 2: Constructed quintessence potentials $V(\phi)$.

In the evaluation of these equations, we have also chosen the initial values of the quintessence field $\tilde{\phi}_0 = 0.8$ at the redshift $z = 0$ (present). The value of $\tilde{\phi}_0$ is chosen for the purpose of definiteness. If we shift its value, it simply results in the shift of the value of the scalar field; the potential in Fig. 2 is shifted horizontally. It has no influence on the evolution of the universe and the shape of the quintessence potential. In general, ϕ decreases monotonously as z increases from -1 , and the potential increases. This means that the potential decreases as the universe expands.

As shown in Fig. 2, the four cases possess the same asymptotic behavior for the region $0.4 < \phi < 0.8$. This corresponds to low redshift $0 < z < 1$. We can give the approximate analytic form of the potential, which is of the exponential form:

$$\tilde{V}(\tilde{\phi}) \simeq \frac{1}{2}(1 - w_0) \exp \left[-\sqrt{3(1 + w_0)(1 + r_0)} (\tilde{\phi} - \tilde{\phi}_0) \right]. \quad (24)$$

They differ when z becomes large. In this region, we find that the field ϕ becomes small and the quintessence potential is in the form of power law for the first case

$$\tilde{V}(\tilde{\phi}) \simeq \frac{1}{2}(1 - w_0) \left(\frac{-w_0\sqrt{3r_0}}{2\sqrt{1 + w_0}} \tilde{\phi} \right)^{2(1+w_0)/w_0}, \quad (25)$$

and the potentials are complicated for the last three cases.

Actually it is possible to give an analytic form of the potential in Case I. We first integrate Eq. (17) to obtain

$$\tilde{\phi} = \frac{\sqrt{1 + w_0}}{\sqrt{3}w_0} \ln \frac{[1 + r_0(1 + z)^{-3w_0}]^{1/2} - 1}{[1 + r_0(1 + z)^{-3w_0}]^{1/2} + 1}, \quad (26)$$

Solving this for $1 + z$ and substituting the result into Eq. (16), we obtain

$$\tilde{V}(\tilde{\phi}) = \frac{1}{2}(1 - w_0) \left(\frac{1}{r_0 \sinh^2(\sqrt{3}w_0\tilde{\phi}/2\sqrt{1 + w_0})} \right)^{-(1+w_0)/w_0}. \quad (27)$$

This is of course consistent with the above asymptotic forms (24) and (25). This potential was also given in Ref. [19]. We thus confirm that the potential is indeed a runaway type for $-1 < w_0 < 0$, and its asymptotic value is zero. This is a very interesting result since one would obtain such behavior in general in supersymmetric theories. This kind of potential is also the one expected for tachyons in unstable D-brane system in superstring theories [20].

In conclusion, we have developed a method of constructing the quintessence potential directly from the effective equation of state function $w_\phi(z)$, which describes the properties of the dark energy. Then we have considered four parametrizations of equation of state parameter and showed that the constructed quintessence potential takes the form of a runaway type. The future Supernova/Acceleration Probe with high-redshift observations [21], in combination with the Planck CMB observation [22], will be able to determine the parameters in the dark energy parametrization to high precision. By precision mapping of the recent expansion history, we hope to learn more about the essence of the dark energy.

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